ON THE BEHAVIOR OF THE SOLUTION OF THE FIRST FUNDAMENTAL PROBLEM OF THE THEORY OF ELASTICITY FOR A LONG RECTANGULAR PLATE

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In this work a description of the solution of the first fundamental problem in the theory of elasticity is obtained for a rectangle in the vicinity of corner points. On the basis of this concept a series of conclusions are drawn about the differential properties of the solution as a function of properties of boundary functions, and the final formulation of the fundamental result of paper [1] is pointed out. In conclusion the behavior of the solution is elucidated for the case where the relative width of the rectangle tends to zero.

1. On the rectangle $ABCD$ (Fig. 1) let us examine the first fundamental problem of the theory of elasticity

$$\Delta^2 u = 0 \tag{1.1}$$

$$u = u_y = 0, \quad y = \pm 1; \quad u = f(y), \quad u_x = \pm f_1(y), \quad x = \pm h \tag{1.2}$$

The fundamental objective of this work will be the analysis of the behavior of the solution u of problems (1.1) and (1.2) for the case where the relative width of the plate tends to zero, i.e. for $h \to \infty$. The indicated problem arises in the proof of the applied theory of bending of rods and in the examination of the accuracy of this theory. The solution is based on the special representation (3.22) of paper [1], which was obtained with the assumption that $u \in W_p^4$ $(p > 2)$. The question about the necessity to require that boundary functions $f(y)$ and $f_1(y)$ satisfy the condition $u \in W_p^4$ $(p > 2)$, remained open in [1].

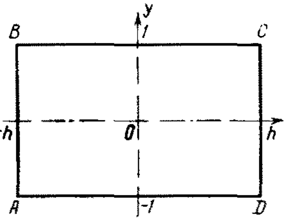

Fig. 1

At present differential properties of solutions of elliptic equations inside the region and near the smooth parts of the boundary have been well studied. Among the efforts devoted to the investigation of solutions in the vicinity of singular points of the boundary the work of Kondrat'ev should be noted [2], where general elliptic equations in regions with conic or corner points are investigated. The remaining papers of this type are either devoted to second order equations, or they impose too harsh limitations on the solutions.

In the present paper an analysis of behavior of the solution of problems (1.1) and (1.2) in the vicinity of corners of rectangle $ABCD$ is carried out, which in spite of the general idea differs from the investigation carried out by Kondrat'ev in [2]. The method which is used by the author was developed by Vorovich who examined the mixed problem of the theory of elasticity for a strip and a layer and who elucidated the character of behavior of the solution of these problems at infinity and at points of separation of boundary conditions (*).

*) See summaries of Transactions of the III All-Union Conference on Theoretical and Applied Mechanics.

2. In this section all auxiliary material is collected for subsequent use.

a) Basic notation: Ω is the rectangle $ABCD$; $\Omega_{R,A}$ is the sector of radius R with the center at point A; $C_{l+\alpha}(D)$ is the space of functions, definite in region D and having in this region lth derivatives which satisfy Hölder's condition with the index α (for $l = 0$ we write C_α); $L_p(a, b)$ is the space of functions which are summable in the pth degree on the section $[a, b]$; $W_p^l(D)$ is the space of functions which in region D have generalized lth derivatives summable in the pth degree [3], H_0 is the set of functions $u \in W_2^2(\Omega)$ and which satisfy the boundary conditions $u|_S = du/dn|_S = 0$, where S is the boundary of the region Ω; $W_p^{l-1/p}(a, b)$ is the space of Slobodetskii [4].

b) The function $u \in W_2^2(\Omega)$ which satisfies the boundary conditions (1.2) and the integral identity

$$\iint_\Omega \Delta u \Delta v dxdy = 0 \quad (v \in H_0) \tag{2.1}$$

is called the generalized solution of the problem (1.1), (1.2).

Lemma 2.1. If the system of boundary values (1.2) is admissible, then there exists a unique function $u \in W_2^2(\Omega)$ which satisfies the problem (1.1), (1.2) in the generalized sense [3].

The problem of admissibility of the system of boundary values in the case of a region with a smooth boundary was completely solved in the work of Slobodetskii [4]. However, in the presence of corner points this question still remains open. For the problem (1.1), (1.2) we can present the following sufficient condition.

Lemma 2.2. If $f(y), f_1(y) \in W_2^2(-1,1)$ and in addition if $f(\pm 1) = f'(\pm 1) = f_1(\pm 1) = f_1'(\pm 1) = 0$, then the system of boundary values (1.2) is admissible.

In the following, however, it will not be assumed that conditions of Lemma 2.2 are satisfied. We shall only assume that the system of boundary values is admissible. It follows then from imbedding theorems of Sobolev-Kondrashov that:

$$f(y) \in C_\alpha(-1,1) \quad (\alpha < 1), \quad f(\pm 1) = 0 \tag{2.2}$$

$$f'(y), \ f_1(y) \in L_p(-1,1) \quad (p < \infty) \tag{2.3}$$

and also

$$u \in C_\alpha(\Omega) \ (\alpha < 1) \tag{2.4}$$

Lemma 2.3. The function u which is the generalized solution of problem (1.1), (1.2) is infinitely differentiable inside Ω (see e.g. [3], p. 117).

Lemma 2.4. If $f \in C_{l+\alpha}(a, b)$, and $f_1 \in C_{l-1+\alpha}(a, b)$, $l \geqslant 1$, $0 < \alpha < 1$, $a > -1$, $b < 1$, then $u \in C_{l+\alpha}(D)$, where D is a closed subregion of Ω, adjoining the interval (a, b) and not containing corner points of Ω. The following estimate is valid:

$$\|u\|_{C_{l+\alpha}} \leqslant C(\|f\|_{C_{l+\alpha}} + \|f_1\|_{C_{l-1+\alpha}})$$

Lemma 2.5. If $f \in W_p^{l-1/p}(a, b)$, and $f_1 \in W_p^{l-1-1/p}(a, b)$, $l \geqslant 2$, $1 < p < \infty$ ($p \geqslant 2$ for $l = 2$), then $u \in W_p^l(D)$ in any closed subregion D of the region Ω, adjoining the interval (a, b) and not containing corner points of Ω. The following estimate is valid:

$$\|u\|_{W_p^l} \leqslant C(\|f\|_{W_p^{l-1/p}} + \|f_1\|_{W_p^{l-1-1/p}})$$

Lemmas 2.4 and 2.5 follow from results of work in [5].

c) The following integral is called the Mellin transform of function $f(r)$

$$F(s) = \int_0^\infty r^{s-1} f(r) dr, \quad s = \sigma + i\tau$$

Lemma 2.6. If $r^{\sigma-1} f(r) \in L_1(0, \infty)$, where function $f(r)$ is continuous and has bounded variation, then

$$f(r)=\frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} r^{-s}F(s)ds$$

where the integral is undestood in the sense of the principal value [6].

Lemma 2.7. If $f(r)\in L_1(a, b)$ and $f(r)\equiv 0$ outside of $[a, b]$ $(a>0,\ b<\infty)$, then $F(s)$ is an entire function.

Lemma 2.8. If $f(r)\equiv 0$ for $r\geqslant a$ and $f(r)\in L_p(0, a)$, then $F(s)$ is an analytic function in the half-plane $\sigma>1/p$.

Lemma 2.9. If $f(r)\equiv 0$ outside of $[a, b]$ $(a>0,\ b<\infty)$ and $f(r)\in C_{l+\alpha}(0,\infty)$, then for large τ $\quad |F(s)|\leqslant C\|f\|_{C_{l+\alpha}}|s|^{-l}|e^{\pi/|\tau|}-1|^{\alpha}$

Lemma 2.10. If $f(r)\in C_{l+\alpha}(0,\infty)$ and $f(r)\equiv 0$ for $r\geqslant a>0$, then $F(s)$ is an analytic function in the half-space $\sigma>-(l+\alpha)$, with the exception of perhaps the points $s=0, -1,\dots,-l$, where simple poles are possible. For large τ the following estimate is valid: $|F(s)|\leqslant C(\gamma)\|f\|_{C_{l+\alpha}}|s|^{-l}|e^{\pi/|\tau|}-1|^{\gamma}$ $\quad (\gamma=\min(\alpha,\ \sigma+l+\alpha))$

Lemmas 2.7–2.10 are taken from still unpublished work of I. I. Vorovich.

3. The integral identity (2.1) for any four times continuously differentiable function $v\in H_0$ can be rewritten in the following manner:

$$\iint_{\Omega} u\Delta^2 v\,dxdy+\int_{-1}^{1}\left[f_1(y)\Delta v\Big|_{x=h}-f(y)\frac{\partial(\Delta v)}{\partial x}\Big|_{x=h}\right]dy+$$

$$+\int_{-1}^{1}\left[f_1(y)\Delta v\Big|_{x=-h}+f(y)\frac{\partial(\Delta v)}{\partial x}\Big|_{x=-h}\right]dy=0 \tag{3.1}$$

It is quite evident that the identity (3.1) is valid for a wider classs of functions v, namely for $v\in H_0\cap W_p^4(\Omega)$ $(p>1)$. In (3.1) let us change to polar coordinates placing their origin at point A. The function v is taken in the form

$$v=r^{s+2}\chi(r)\,a(\varphi) \tag{3.2}$$

where $\chi(r)\equiv 1$ for $r\leqslant R-\delta$ $(0<R<1,\ \ 0<\delta<R)$, $\chi(r)\equiv 0$ for $r\geqslant R$ and $\chi(r)$ is infinitely differentiable for $0\leqslant r<\infty$, and $a(\varphi)$ is four times continuously differentiable. For the function v of form (3.2) to belong to $H_0\cap W_p^4(\Omega)$, it is necessary for the following conditions to be fulfilled: $\sigma>0$ $(\sigma=\mathrm{Re}\,s)$, $a(0)=a'(0)=$ $=a(^1/_2\pi)=a'(^1/_2\pi)=0$. The identity (3.1) now assumes the following form:

$$\iint_{\Omega_{R,A}} u\{r^{s-2}\chi(r)[a^{\mathrm{IV}}+(s^2+(s+2)^2)a''+s^2(s+2)^2a]+r^{s-1}\chi'(r)[(4s+6)a''+$$

$$+(4s^3+18s^2+24s+9)a]+r^s\chi''(r)[2a''+(6s^2+24s+23)a]+r^{s+1}\chi'''(r)(4s+10)a+$$

$$+r^{s+2}\chi^{\mathrm{IV}}(r)a\}\,rdrd\varphi+\int_0^R[f_1(r-1)r^s\chi(r)a''(^1/_2\pi)-f(r-1)r^{s-1}\chi(r)a'''(^1/_2\pi)]dr=0 \tag{3.3}$$

Let us assume

$$U(\varphi,s)=\int_0^R u(r,\varphi)\chi(r)r^{s-1}dr \tag{3.4}$$

$$U_k(\varphi,s)=\int_{R-\delta}^{R} u(r,\varphi)\chi^{(k)}(r)r^{s-1+k}dr \quad (k=1,\ 2,\ 3,\ 4) \tag{3.5}$$

$$F(s)=\int_0^R f(r-1)\chi(r)r^{s-1}dr \tag{3.6}$$

$$F_1(s)=\int_0^R f_1(r-1)\chi(r)r^{s}dr \tag{3.7}$$

Apparently, U (φ, s) will be Mellin's transform of function $u(r,\varphi)\chi(r)$. It follows from (2.4) that $U(\varphi, s)$ and $U_k(\varphi, s)$ on $[0, {}^1/_2\pi]$ will be continuous functions of variable φ. From Lemma 2.3 it follows that $U_k(\varphi, s)$ are infinitely differentiable for $\varphi < {}^1/_2\pi$.

Substituting (3.4) to (3.7) into (3.3) we obtain

$$\int_0^{1/2\pi}\{U[a^{IV}+(s^2+(s+2)^2)a''+s^2(s+2)^2a]+U_1[(4s+6)a''+(4s^3+18s^2+ \\ +24s+9)a]+U_2[2a''+(6s^2+24s+23)a]+ \\ +U_3(4s+10)a+U_4a\}d\varphi+F_1(s)+a''({}^1/_2\pi)-F(s)\,a'''({}^1/_2\pi)=0 \tag{3.8}$$

Let us introduce the operators

$$KU=\int_\varphi^{{}^1/_2\pi}(\vartheta-\varphi)U(\vartheta)\,d\vartheta,\qquad K_1U=\int_\varphi^{{}^1/_2\pi}\frac{(\vartheta-\varphi)^3}{3!}U(\vartheta)\,d\vartheta$$

It is easy to check that for any four times continuously differentiable function $a(\varphi)$ which statisfies the conditions $a(0)=a'(0)=a''(0)=a'''(0)=0$ the following relationship holds

$$\int_0^{{}^1/_2\pi}Ua''d\varphi=\int_0^{{}^1/_2\pi}KUa^{IV}\,d\varphi,\qquad \int_0^{{}^1/_2\pi}Ua\,d\varphi=\int_0^{{}^1/_2\pi}K_1Ua^{IV}d\varphi \tag{3.9}$$

It is also obvious that

$$F_1(s)a''({}^1/_2\pi)-F(s)\,a'''({}^1/_2\pi)=\int_0^{{}^1/_2\pi}[{}^1/_2\pi F_1(s)-F(s)-\varphi F_1(s)]\,a^{IV}d\varphi \tag{3.10}$$

Placing on function $a(\varphi)$ in (3.8) the new constraint $a''(0)=a'''(0)=0$ and utilizing (3.9) and (3.10), we obtain

$$\int_0^{{}^1/_2\pi}\{U+K[(s^2+(s+2)^2)U+(4s+6)U_1+2U_2]+K_1[s^2(s+2)^2U+ \\ +(4s^3+18s^2+24s+9)U_1+(6s^2+24s+23)U_2+ \\ +(4s+10)U_3+U_4]+{}^1/_2\pi F_1(s)-F(s)-\varphi F_1(s)\}\,a^{IV}(\varphi)d\varphi=0 \tag{3.11}$$

Since the expression in braces in identity (3.11) will be a continuous function of variable φ, while $a^{IV}(\varphi)$ is an arbitrary continuous function, then according to the fundamental lemma of variational calculus it follows from (3.11) that

$$U(\varphi,s)=-K[(s^2+(s+2)^2U+(4s+6)U_1+2U_2]-K_1[s^2(s+2)^2U+ \\ +(4s^3+18s^2+24s+9)U_1+(6s^2+24s+23)U_2+(4s+10)U_3+U_4]+ \\ +F(s)-{}^1/_2\pi F_1(s)+\varphi F_1(s)\quad (0\leqslant\varphi\leqslant{}^1/_2\pi) \tag{3.12}$$

The right side of identity (3.12) has two continuous derivatives. Differentiating twice we find

$$U''=-(s^2+(s+2)^2)U-(4s+6)U_1-2U_2-K[s^2(s+2)^2U+(4s^3+18s^2+ \\ +24s+9)U_1+(6s^2+24s+23)U_2+(4s+10)U_3+U_4] \tag{3.13}$$

The function U is twice continuously differentiable in $[0, {}^1/_2\pi]$, while U_k are infinitely differentiable for $\varphi < {}^1/_2\,\pi$. Therefore for $\varphi < {}^1/_2\,\pi$, Eq. (3.13) can be differentiated two more times. As a result we obtain

$$U^{IV} + (s^2 + (s+2)^2)U'' + s^2(s+2)^2 U = M(\varphi, s) \qquad (3.14)$$

Here

$$M(\varphi, s) = -[(4s+6)U''_1 + 2U_2'' + (4s^3 + 18s^2 + 24s + 9)U_1 + (6s^2 + 24s + 23) \times$$
$$\times U_2 + (4s+10)U_3 + U_4] \qquad (3.15)$$

Taking $\varphi = {}^1/_2\,\pi$ in (3.12) we obtain

$$U(\varphi, s)\,|_{\varphi = {}^1/_2\pi} = F(s) \qquad (3.16)$$

Differentiating (3.12) and putting $\varphi = {}^1/_2\pi$, we find

$$U'(\varphi, s)\,|_{\varphi={}^1/_2\pi} = F_1(s) \qquad (3.17)$$

Boundary conditions for $U(\varphi, s)$ for $\varphi = 0$ are obtained from (3.4)

$$U\,|_{\varphi=0} = U'|_{\varphi=0} = 0 \qquad (3.18)$$

Thus, Mellin's transform of function $u(r, \varphi)\chi(r)$ satisfies Eq. (3.14) and boundary conditions (3.16) – (3.18).

4. Let $G(\varphi, \psi, s)$ be the Green's function of the differential operator determined by Eq. (3.14) and the boundary conditions

$$U\,|_{\varphi=0} = U\,|_{\varphi={}^1/_2\pi} = U'|_{\varphi=0} = U'|_{\varphi={}^1/_2\pi} = 0$$

Then the solution of the problem (3.14), (3.16)–(3.18) is written in the form:

$$U(\varphi, s) = \int\limits_0^{{}^1/_2\pi} G(\varphi, \psi, s) M(\psi, s)\, d\psi + G_{\psi^3}'''(\varphi, {}^1/_2\pi, s) F(s) - G_{\psi^2}''(\varphi, {}^1/_2\pi, s) F_1(s) \qquad (4.1)$$

Green's function $G(\varphi, \psi, s)$ in the explicit form is given by the equations

$$G(\varphi, \psi, s) = \frac{1}{8s(s+1)(s+2)}\Big\{(s+1)\sin[(s+2)\varphi - s\psi] + \sin(s+1)\frac{\pi}{2} \times$$
$$\times \sin\left[\frac{\pi}{2}s - (s+2)\varphi - s\psi\right] \pm s\sin(s+2)(\varphi-\psi) \mp (s+2)\sin s(\varphi - \psi) +$$
$$+ \frac{a^{(1)}(\varphi, s)\, b^{(1)}(\psi, s)}{D_1(s)} + \frac{a^{(2)}(\varphi, s)\, b^{(2)}(\psi, s)}{D_2(s)}\Big\} \qquad (4.2)$$

$$a^{(1)}(\varphi, s) = (s+2)\sin s(\varphi - {}^1/_4\pi) - \cos {}^1/_2(s+1)\pi\cos[(s+2)\varphi - {}^1/_4 s\pi]$$
$$a^{(2)}(\varphi, s) = (s+2)\cos s(\varphi - {}^1/_4\pi) - \cos {}^1/_2(s+1)\pi\sin[(s+2)\varphi - {}^1/_4 s\pi]$$
$$b^{(1)}(\psi, s) = s\cos[(s+2)\psi - {}^1/_4 s\pi] + \cos {}^1/_2(s+1)\pi\sin s(\psi - {}^1/_4\pi)$$
$$b^{(2)}(\psi, s) = s\sin[(s+2)\psi - {}^1/_4 s\pi] + \cos {}^1/_2(s+1)\pi\cos s(\psi - {}^1/_4\pi)$$
$$D_1(s) = \sin {}^1/_2(s+1)\pi - (s+1), \quad D_2(s) = \sin {}^1/_2(s+1)\pi + (s+1) \qquad (4.3)$$

In (4.2) the upper sign is taken for $\varphi < \psi$, the lower sign for $\varphi > \psi$.

It is evident from (4.2) that $G(\varphi, \psi, s)$ will be a meromorphic function of parameter s. This function will have simple poles in those points in which the denominators $D_1(s)$ and $D_2(s)$ become zero. The points $s = 0, -1, -2$, represent an exception. It is easy to check that these points will not be singular for $G(\varphi, \psi, s)$. Subsequently only those zeros s_k of functions $D_1(s)$ and $D_2(s)$ are of interest for which Re $s_k < 0$ ($1_k \neq 0, -1, -2$). We shall renumber them in the order of increasing moduli, giving the same index k to zeros with equal moduli (as is evident from (4.3), one index k corresponds to two complex conjugate zeros $D_1(s)$ and $D_2(s)$). It is easy to show that

$$s_k = -2 - 2k \pm i{}^1\!/_2\pi \ln 2(1 + 2k) + O(k^{-1}\ln k)$$

In [8] a table of roots is given for equations $\operatorname{sh} z \pm 2z/\pi = 0$. After appropriate transformations it follows from this table that (*)

$$s_1 \approx -3.739 \pm 1.119i, \qquad s_2 \approx -5.808 \pm 1.464i, \qquad s_3 \approx -7.843 \pm 1.681i, \ldots \tag{4.4}$$

Utilizing the explicit representation (4.2) of the Green function, we can verify directly that

$$|G(\varphi, \psi, s)| \leqslant C|s|^{-1}, \; |G''_{\psi_2}(\varphi, {}^1\!/_2\pi, s)| \leqslant C, \; |G'''_{\psi_3}(\varphi, {}^1\!/_2\pi, s)| \leqslant C|s| \tag{4.5}$$

applies in the entire s-plane with the exception of regions near the poles.

5. The representation (4.1) of function $U(\varphi, s)$ was obtained for $\sigma > 0$; however, it also applies for $\sigma > -1$ by virtue of analyticity of the right side of (4.1) (analyticity follows from (2.2), (2.3), Lemmas 2.7, 2.8, 2.10 and properties of Green's function). Let us apply the inverse Mellin transform to (4.1). Considering $r < R - \delta$, we obtain

$$u(r, \varphi) = \frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} r^{-s} U(\varphi, s)\, ds \tag{5.1}$$

Here by virtue of Lemma 2.6 we can write $\sigma = -1 + \delta$, where $\delta > 0$ can be taken arbitrarily small.

It follows from (5.1) that $u(r, \varphi)$ cannot have singularities r^s, where $\operatorname{Re} s < 1$, more exactly for $\operatorname{Re} s < 1$

$$\lim_{r \to 0} u(r, \varphi) r^{-s} = 0$$

In the derivation of Eq. (5.1), only the fact that function u belongs to space $W_2^2(\Omega)$ and the consequences of this property (2.2)–(2.4) were used. Let us assume now that $f \in C_{l+\alpha}(-1, R-1)$, $f_1 \in C_{l-1+\alpha}(-1, R-1)$, $l \geqslant 2$, $0 < \alpha < 1$. Then, according to Lemma 2.9 (see (3.15), (3.5) and Lemma 2.4) $M(\psi, s)$ is an entire function and the following estimate is valid:

$$|M(\psi, s)| \leqslant C(\|f\|_{C_{l+\alpha}} + \|f_1\|_{C_{l-1+\alpha}})|s|^{-l+3}|e^{\pi/|\tau|} - 1|^{\alpha} \tag{5.2}$$

According to Lemma 2.10, $F(s)$ and $F_1(s)$ are analytic in the half-plane $\sigma > -(l + \alpha)$ with perhaps the exception of points $s = -1, -2, \ldots, -l$. The following estimates are valid:

$$|F(s)| \leqslant C\|f\|_{C_{l+\alpha}}|s|^{-l}|e^{\pi/|\tau|} - 1|^{\gamma}$$

$$|F_1(s)| \leqslant C\|f_1\|_{C_{l-1+\alpha}}|s|^{-l+1}|e^{\pi/|\tau|} - 1|^{\gamma} \tag{5.3}$$

It follows from this that $U(\varphi, s)$ is analytically extendable into the strip $-(l + \alpha) < \sigma \leqslant -1$ and has perhaps in this strip simple poles of points $s = s_k$ and $s = -k$ $(k = 1, 2, \ldots, l)$. The following estimate results from (5.2), (5.3) and (4.5)

$$|U(\varphi, s)| \leqslant C(\|f\|_{C_{l+\alpha}} + \|f_1\|_{C_{l-1+\alpha}})|s|^{-l+2}|e^{\pi/|\tau|} - 1|^{\gamma} \tag{5.4}$$

Analyticity of $U(\varphi, s)$ and estimate (5.4) make it possible in (5.1) to pass from integration along the straight line $\sigma = -1 + \delta$ to integration along the straight line $\sigma = \sigma_{l+\alpha} = -(l + \alpha) + \delta$. In this connection we obtain

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*) The statement of Kondrat'ev [2] (p. 289) that in the strip $1 < \operatorname{Re} z < 2$ there is a root of the equation $\sin^2 {}^1\!/_2\pi z - z^2 = 0$ is erroneous. The derivation made on the basis of this statement is also incorrect.

$$u(r, \varphi) = \frac{1}{2\pi i} \int\limits_{\sigma_{l+\alpha}-i\infty}^{\sigma_{l+\alpha}+i\infty} r^{-s} U(\varphi, s)\, ds + \Sigma \operatorname{res} r^{-s} U(\varphi, s) \quad (5.5)$$

Computations in (5. 5) must be made with respect to all poles of the function under the integral which lie in the strip $\sigma_{l+\alpha} < \sigma < -1 + \delta$. We have

$$\operatorname{res} r^{-s} U(\varphi, s)|_{s=-k} = r^k A_k(\varphi) \quad (5.6)$$

Here

$$A_1(\varphi) = \frac{2}{\pi^2 - 4}\{[2\varphi\cos\varphi + (\pi\varphi - 2)\sin\varphi] f'(0) - [\pi\varphi\cos\varphi + (2\varphi - \pi)\sin\varphi] f_1(0)\}$$

$$A_2(\varphi) = {}^1/_2 [\sin^2\varphi\, f''(0) - ({}^1/_2\pi \sin^2\varphi + \sin\varphi\cos\varphi - \varphi) f_1'(0)]$$

$$A_{2n+1}(\varphi) = \frac{(-1)^n}{(2n+1)!}\left[\left(\cos 2n\varphi \sin\varphi - \frac{1}{2n}\sin 2n\varphi\cos\varphi\right) f^{(2n+1)}(0) + \left(1 + \frac{1}{2n}\right)\sin 2n\varphi \sin\varphi\, f_1^{(2n)}(0)\right] \quad (n = 1, 2, \ldots)$$

$$A_{2n+2}(\varphi) = \frac{(-1)^n}{(2n+2)!}\left\{\sin(2n+1)\varphi\sin\varphi\, f^{(2n+2)}(0) - \left[\left(1 + \frac{1}{2n}\right)\cos(2n+1)\varphi\sin\varphi - \frac{1}{2n}\sin(2n+1)\varphi\cos\varphi\right] f_1^{(2n+1)}(0)\right\} \quad (n = 1, 2, \ldots)$$

$$\operatorname{res} r^{-s} U(\varphi, s)|_{s=s_k} = c_k r^{-s_k} a_k(\varphi) \quad (5.7)$$

where

$$c_k = \frac{1}{\gamma_k}\left[\sin {}^1/_4 s_k\pi\, F(s_k) + \frac{\cos {}^1/_4 s_k\pi}{s_k + 2} F_1(s_k) - \frac{1}{s_k(s_k+1)(s_k+2)} \int\limits_0^{{}^1/_2\pi} M(\psi, s_k)\, b_k(\psi)\, d\psi\right] \quad (k = 2n)$$

$$c_k = \frac{1}{\gamma_k}\left[-\cos {}^1/_4 s_k\pi F(s_k) + \frac{\sin {}^1/_4 s_k\pi}{s_k + 2} F_1(s_k) - \frac{1}{s_k(s_k+1)(s_k+2)} \int\limits_0^{{}^1/_2\pi} M(\psi, s_k)\, b_k(\psi)\, d\psi\right] \quad (k = 2n - 1)$$

$a_k(\varphi) = a^{(1)}(\varphi, s_k)$, $b_k(\psi) = b^{(1)}(\psi, s_k)$, $\gamma_k = \pi \sin {}^1/_2 s_k\pi + 2$ $(k = 2n)$

$a_k(\varphi) = a^{(2)}(\varphi, s_k)$, $b_k(\psi) = b^{(2)}(\psi, s_k)$, $\gamma_k = \pi \sin {}^1/_2 s_k\pi - 2$ $(k = 2n - 1)$

Substituting (5.6) and (5.7) into (5.5) we finally obtain

$$u(r, \varphi) = \sum_{k=1}^{l} r^k A_k(\varphi) + \sum_{\operatorname{Re} s_k > \sigma_{l+\alpha}} c_k r^{-s_k} a_k(\varphi) + O(r^{(l+\alpha)-\delta}) \quad (5.8)$$

As is evident from estimate (5.4), Eq. (5.8) can be differentiated with respect to both variables $l - 3$ times. In this manner the theorem is proved.

Theorem 5.1. If $f \in C_{l+\alpha}(-1, R - 1)$, $f_1 \in C_{l-1+\alpha}(-1, R - 1)$, $l \geqslant 2$, $0 < \alpha < 1$, then function u, which is the generalized solution of problem (1.1), (1.2), in the vicinity of point A has an $l - 3$ times differentiable representation of the form (5.8).

The representation (5.8) permits to draw some conclusions about the differential properties of function u in the vicinity of corner points. For example:

a) If $f \in C_{4+\alpha}(-1, R-1)$, $f_1 \in C_{3+\alpha}(-1, R-1)$ and $f'(-1) = f_1(-1) = 0$, then

$$u(x, y) \in C_{1+\beta}(\Omega_{R-\varepsilon, A}) \quad (0 < \beta \leqslant 1);$$

b) If $f \in C_{5+\alpha}(-1, R-1)$, $f_1 \in C_{4+\alpha}(-1, R-1)$ and $f'(-1) = f''(-1) = f_1(-1) = f_1'(-1) = 0$, then

$$u(x, y) \in C_{2+\beta}(\Omega_{R-\varepsilon, A}) \quad (0 < \beta \leqslant 1)$$

c) If $f \in C_{6+\alpha}(-1, R-1)$, $f_1 \in C_{5+\alpha}(-1, R-1)$ and $f'(-1) = f''(-1) = f'''(-1) = f_1(-1) = f_1'(-1) = f_1''(-1) = 0$, then

$$u(x, y) \in C_{3+\beta}(\Omega_{R-\varepsilon, A}) \quad (0 < \beta \leqslant \approx 0.739)$$

d) If $f \in C_{7+\alpha}(-1, R-1)$, $f_1 \in C_{6+\alpha}(-1, R-1)$ and $f'(-1) = f''(-1) = f'''(-1) = f_1(-1) = f_1'(-1) = f_1''(-1) = 0$, then

$$u(x, y) \in W_p^4(\Omega_{R-\varepsilon, A}) \qquad (p \leqslant \approx 7.66)$$

For further improvement of differential properties of the function u in the vicinity of point A it is necessary to require that not only the corresponding derivatives of functions f and f_1 become zero at the point -1, but also some first coefficients c_k.

Statement (d) and Lemmas 2.2 and 2.5 permit to formulate the principal result of paper [1] in the following closed form.

Theorem 5.2. If $f \in W_p^{4-1/p}(-1,1) \cap C_{7+\alpha}(-1, R-1) \cap C_{7+\alpha}(1-R, 1)$, $f_1 \in W_p^{3-1/p}(-1,1) \cap C_{6+\alpha}(-1, R-1) \cap C_{6+\alpha}(1-R, 1)$, $p > 2$, $0 < \alpha < 1$ and $f(\pm 1) = f'(\pm 1) = f''(\pm 1) = f'''(\pm 1) = f_1(\pm 1) = f_1'(\pm 1) = f_1''(\pm 1) = 0$, then the solutio u of problem (1.1), (1.2) is uniquely representable in the form

$$u(x, y) = \sum_{k=1}^{\infty} [c_k^{(1)} a_k^{(1)}(y) \cos \lambda_k^{(1)} x + c_k^{(2)} a_k^{(2)}(y) \cos \lambda_k^{(2)} x] \tag{5.9}$$

6. The representation of the function $u(x, y)$ in the form of a series (5.9) makes it possible to analyze the behavior of the function when the relative width of the rectangle $ABCD$ tends to zero, i.e. for $h \to \infty$.

Theorem 6.1. If conditions of Theorem 5.2 are fulfilled, then function $u(x, y)$, which is a solution of problem (1.1), (1.2), disappears uniformly together with all derivatives for $h \to \infty$ in any bounded subregion of the rectangle $ABCD$.

In the representation (5.9), as was shown in [1] $c_k = O(e^{-k\pi h} k^{-5/2})$, $a_k(y) = O(k^{3/2})$ and $\cos \lambda_k x = O(e^{k\pi |x|})$, so that if $h \to \infty$, and x remains bounded, then each term of series (5.9) decreases exponentially, which proves Theorem 6.1.

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ON THE METHOD OF ORTHOGONAL POLYNOMIALS IN CONTACT PROBLEMS OF THE THEORY OF ELASTICITY

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It is shown that the application of orthogonal polynomials to contact problems [1−12] is closely associated with the existence of a certain special calss of so-called polynomial kernels [13]. In [4, 6, 14, 15] particular cases of such kernels were constructed in various ways. Here we indicate a method of constructing polynomial kernels, on the basis of which not only all the previously constructed kernels may be obtained, but more general ones as well.

1. The essential features of the method of orthogonal polynomials. It is known that spatial contact problems with no friction force may be reduced to a two-dimensional integral equation of the first kind. To this equation must be adjoined a differential equation as well, if a plate rather than a stamp is being contacted. For contact regions such as a half-plane, strip, disk, or annulus, by means of some integral transformation or another, one may reduce the indicated two-dimensional system of equations to a one-dimensional problem. In the case of a stamp we thus have only a single one-dimensional integral equation of the first kind. In the case of a plate, however, we obtain a system composed of the equation indicated together with an ordinary differential equation. This last can likewise be reduced to an integral equation of the first kind by use of the Green's function for the differential equation obtained. One may get an idea of how this is done by looking at the example of a plane contact problem in [12].

Thus, spatial contact problems for the regions enumerated, and also plane problems with one contacting segment (sometimes two) may be reduced to solving an integral equation of the first kind

$$\int_a^b K(x, y)\varphi(y)\,dy = f(x) \qquad (a \leqslant x \leqslant b) \tag{1.1}$$

given on either a finite or a semi-infinite interval.

Such problems, but with account taken of the surface structure of the contacting bodies, were, in the formulation of Shtaerman [16], reduced to analogous integral equations of the second kind